

**WEIGHTED THEOREMS ON FRACTIONAL INTEGRALS  
 IN THE GENERALIZED HÖLDER SPACES  
 VIA INDICES  $m_\omega$  AND  $M_\omega$**

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**Abstract**

There are known various statements on weighted action of one-dimensional and multidimensional fractional integration operators in spaces of continuous functions, such as weighted generalized Hölder spaces  $H_0^\omega(\rho)$  of functions with a given dominant  $\omega$  of their continuity modulus. Conditions under which the fractional integration operator maps the space  $H_0^\omega(\rho)$  onto the better space  $H_0^{\omega_\alpha}(\rho)$  with  $\omega_\alpha(h) = h^\alpha \omega(h)$ , were given in terms of certain integral conditions on the weight function  $\rho$  and the characteristic  $\omega$  (Zygmund type conditions). In this paper all the known results of such kind are reconsidered and obtained in the explicit form of inequalities involving the order  $\alpha$  of fractional integration, certain numerical characteristics of the weight function  $\rho$  and the so called upper and lower indices  $m_\omega$  and  $M_\omega$  of  $\omega(h)$ . We prove a theorem providing the equivalence of the integral Zygmund conditions to some direct numerical inequalities for the indices  $m_\omega$  and  $M_\omega$ . Based on that theorem we prove a series of new theorems on action of fractional integrals in the generalized Hölder spaces  $H_0^\omega(\rho)$  for various types of weights, both in one-dimensional and multi-dimensional cases.

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## 1. Introduction

Let  $\operatorname{Re} \alpha > 0$ . Statements on weighted action of the Riemann-Liouville fractional operators

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^{1-\alpha}}, \quad x > a, \quad (1.1)$$

or Liouville fractional operators

$$I_{+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(t) dt}{(x-t)^{1-\alpha}}, \quad x \in \mathbb{R}^1, \quad (1.2)$$

in the Lebesgue spaces of integrable functions are well known and obtained in a large generality, including necessary and sufficient conditions. It suffices to remind the one-dimensional weighted Hardy-Littlewood theorem [6], the multidimensional weighted Sobolev-Stein-Weiss theorem [27] for power weights, the Muckenhoupt-Wheeden weighted theorem [14], see the books [3], [4], [10], [24] where there also may be found more general operators and more general spaces of integrable functions, including Orlich spaces.

Theorems on action of fractional integrals in the spaces of continuous functions, such as weighted Hölder spaces or generalized Hölder spaces are less known. While a non-weighted statement on action of the fractional integral operator from  $H_0^{\beta}$  into  $H_0^{\beta+\alpha}$  is due to Hardy and Littlewood ([6], see [24], Theorems 3.1 and 3.2), the weighted results [19], [20] with power weights were obtained much later, see their presentation in [24], Theorems 3.3, 3.4 and 13.13). For generalized Hölder spaces  $H_0^{\omega}(\rho)$  of functions with a given dominant of their continuity modulus, statements on mapping properties in the case of power weight were obtained in [17], [16], [25], see also their presentation in [24], Section 13.6. A different proof was suggested in [8], where the case of complex fractional orders was also considered.

The case of weights more general than power ones, including in particular power-logarithmic type weights, in the spaces  $H_0^{\omega}(\rho)$  was considered in [26], where operators more general than just fractional integrals were treated.

Multidimensional fractional type integrals

$$I^{\alpha} f(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y) dy}{|x-y|^{n-\alpha}}, \quad x \in \mathbb{R}^n, \quad 0 < \operatorname{Re} \alpha < n \quad (1.3)$$

(the Riesz potential operators;  $\gamma_n(\alpha)$  is the well known normalizing constant, see [24], p. 490, or [23], p.37) and

$$\mathcal{I}^\alpha f(x) = \frac{1}{\gamma_{n-1}(\alpha)} \int_{\mathbb{S}^{n-1}} \frac{f(s) ds}{|s - \sigma|^{n-1-\alpha}}, \quad \sigma \in \mathbb{S}^{n-1}, \quad 0 < \operatorname{Re} \alpha < n-1 \quad (1.4)$$

(the spherical Riesz potential operator, see [23], p.151) within the frameworks of generalized Hölder spaces  $H_0^\omega(\mathbb{R}^n, \rho)$  or  $H_0^\omega(\mathbb{S}^{n-1}, \rho)$ , respectively, were considered in [28], [30], [31], [32].

Conditions under which the fractional integration operator maps the space  $H_0^\omega(\rho)$  onto the better space  $H_0^{\omega_\alpha}(\rho)$  with  $\omega_\alpha(x) = x^\alpha \omega(x)$ , in all the above cited papers were given in terms of certain integral conditions on the weight function  $\rho$  and the characteristic  $\omega$  (Zygmund type conditions).

Meanwhile, for functions  $\omega$  in the Zygmund-Bari-Stechkin class, there exist some numerical characteristics, the so called indices, similar to indices known as the Boyd indices or Orlicz-Matuszewska indices in the theory of Orlicz spaces, see [13], p. 20; [11], p. 75; [12]; or [2], p. 149. The indices  $m_\omega$  and  $M_\omega$  of functions  $\omega$  characterizing the generalized Hölder spaces  $H_0^\omega(\rho)$  in the form appropriate for our goals were introduced in [21], [22].

A natural question is whether it is possible to obtain results on mapping properties of fractional integration operators in generalized Hölder spaces not in the "indirect" terms of integral conditions of Zygmund type, but in terms of the direct numerical interval for the exponents of the weight, with boundaries depending on indices  $m_\omega$  and  $M_\omega$ . (In case of non power weights similar indices of weight functions should be also used). This is the goal of this paper to obtain such results on mapping properties of fractional integrals in terms of information about the indices  $m_\omega$  and  $M_\omega$ . In fact, we undertake a revision, in the language of the indices  $m_\omega$  and  $M_\omega$ , of the known statements on mapping properties of fractional type operators in the Hölder type spaces  $H_0^\omega(\rho)$ .

To this end, in Section 3 we prove the principal Theorem 3.5 which provides the equivalence of the integral Zygmund conditions to some direct inequalities for the indices  $m_\omega$  and  $M_\omega$ . To obtain Theorem 3.5, we first prove crucial auxiliary statements (see Theorems 3.1 and 3.2) on equivalence of the so called Bari, Lozinski and Stechkin conditions to the Zygmund conditions on almost increasing functions.

Basing on Theorem 3.5, in Sections 4 and 5 we prove a series of theorems on action of fractional integrals in the generalized Hölder spaces  $H_0^\omega(\rho)$  for various types of weights, both in one-dimensional and multi-dimensional cases.

## 2. Preliminaries

### 2.1. Index numbers $m_\omega$ and $M_\omega$ of functions $\omega \in W$

We need the following definitions.

A non-negative function  $\varphi$  on  $[0, \ell]$  is said to be almost increasing (or almost decreasing) if there exists a constant  $C \geq 1$  such that  $\varphi(x) \leq C\varphi(y)$  for all  $x \leq y$  (or  $x \geq y$ , respectively). Let

$$W = \{\varphi \in C([0, \ell]) : \varphi(0) = 0, \varphi(x) > 0 \text{ for } x > 0, \quad (2.1)$$

$$\varphi(x) \text{ is almost increasing}\}.$$

DEFINITION 2.1. Let  $\omega \in W$ . The numbers

$$m_\omega = \sup_{x>1} \frac{\ln \left[ \lim_{h \rightarrow 0} \frac{\omega(xh)}{\omega(h)} \right]}{\ln x}, \quad M_\omega = \inf_{x>1} \frac{\ln \left[ \overline{\lim}_{h \rightarrow 0} \frac{\omega(xh)}{\omega(h)} \right]}{\ln x},$$

introduced in such a form in [21], [22], will be referred to as the *lower and upper index numbers* of a function  $\omega(x) \in W$  (compare these indices with the Matuszewska-Orlicz indices, see [13], p. 20; they are of the type of the Boyd indices, see [11], p. 75; [12], or [2], p. 149 about the Boyd indices).

For  $\omega \in W$  we have  $0 \leq m_\omega \leq M_\omega \leq \infty$ .

We call a characteristic  $\omega(x)$  *equilibrated* or *non-oscillating*, if  $M_\omega = m_\omega$ .

DEFINITION 2.2. ([1], [5]) The Zygmund-Bari-Stechkin class  $\Phi$  is defined as the class of functions  $\omega \in W$  satisfying the Zygmund conditions

$$\int_0^h \frac{\omega(x)}{x} dx \leq c\omega(h) \quad \text{and} \quad \int_h^\ell \frac{\omega(x)}{x^2} dx \leq c \frac{\omega(h)}{h}, \quad (2.2)$$

where  $c = c(\omega) > 0$  does not depend on  $h \in (0, \ell]$ .

The class  $\Phi$  was introduced in [1], where conditions (2.2) were imposed on monotonic functions in  $W$ ; we deal with almost monotonic functions. The following statement characterizes the class  $\Phi$  in terms of the indices  $m_\omega$  and  $M_\omega$ , see its proof in [22], p. 125 (see also [21]).

THEOREM 2.3. A function  $\omega(x) \in W([0, \ell])$  is in the Bari-Stechkin class  $\Phi$  if and only if

$$0 < m_\omega \leq M_\omega < 1, \quad (2.3)$$

and for  $\omega \in \Phi$  and any  $\varepsilon > 0$  there exist constants  $c_1 = c_1(\omega, \varepsilon) > 0$  and  $c_2 = c_2(\omega, \varepsilon) > 0$  such that

$$c_1 x^{M_\omega + \varepsilon} \leq \omega(x) \leq c_2 x^{m_\omega - \varepsilon}, \quad 0 \leq x \leq \ell. \quad (2.4)$$

Besides this, condition  $m_\omega > 0$  is equivalent to the first inequality in (2.2), while condition  $M_\omega < 1$  is equivalent to the second one.

## 2.2. The generalized Bari-Steckin class $\Phi_\gamma^\beta$

Let  $\beta \geq 0$ ,  $\gamma > 0$ . The following classes  $\Phi_\gamma^\beta$ , considered in [17] (see also [16], [25] and [24], p. 253), generalize the class  $\Phi_\gamma^0$ , introduced in [1]. (Observe that in [34], [35] there were considered more general classes  $\Phi_{b(x)}^{a(x)}$  with limits which may "oscillate"; the classes we deal now correspond to the case where  $a(x) = x^\beta$  and  $b(x) = x^\gamma$ ).

DEFINITION 2.4. The Bari-Steckin type class  $\Phi_\gamma^\beta$  is defined as  $\Phi_\gamma^\beta := \mathcal{Z}^\beta \cap \mathcal{Z}_\gamma$ , where  $\mathcal{Z}^\beta$  is the class of functions  $\omega \in W$  satisfying the condition

$$\int_0^h \frac{\omega(x)}{x^{1+\beta}} dx \leq c \frac{\omega(h)}{h^\beta} \quad (\mathbb{Z}^\beta)$$

and  $\mathcal{Z}_\gamma$  is the class of functions  $\omega \in W$  satisfying the condition

$$\int_h^\ell \frac{\omega(x)}{x^{1+\gamma}} dx \leq c \frac{\omega(h)}{h^\gamma}, \quad (\mathbb{Z}_\gamma)$$

where  $c = c(\omega) > 0$  does not depend on  $h \in (0, \ell]$ .

In the sequel we refer to the above conditions as  $(\mathbb{Z}^\beta)$ - and  $(\mathbb{Z}_\gamma)$ -conditions. The class  $\Phi_\gamma^\beta$  is nonempty if and only if  $\beta < \gamma$ , see Corollary 3.4 below.

Obviously,  $\Phi = \Phi_1^0$  and  $\Phi_\gamma^\beta \subseteq \Phi$  in the case  $0 \leq \beta < \gamma \leq 1$ .

Similarly to (2.4), the class  $\Phi_\gamma^\beta$  is described by the condition  $\beta < m_\omega \leq M_\omega < \gamma$ , which will be proved in Theorem 3.5. The inequalities  $\beta < m_\omega \leq M_\omega < \gamma$  follow immediately from Theorem 2.3, if one knows beforehand that  $\frac{\omega(x)}{x^\beta}$  and  $\frac{\omega(x)}{x^{\gamma-1}}$  both belong to  $W$ , because the lower index of the function  $\frac{\omega(x)}{x^\beta}$  is  $m_\omega - \beta$  and the upper index of the function  $\frac{\omega(x)}{x^{\gamma-1}}$  is  $M_\omega - \gamma + 1$ . The point, however, is that this should be proved for an arbitrary  $\omega \in W$ .

We note that each of the inequalities  $(\mathbb{Z}^\beta)$  and  $(\mathbb{Z}_\gamma)$  is invertible if they both are satisfied. Namely, the following statement holds, in which the equivalence  $f \sim g$  for non-negative functions  $f$  and  $g$  means that there exist positive constants  $C_1$  and  $C_2$  such that  $C_1 f(x) \sim g(x) \sim C_2 f(x)$ .

LEMMA 2.5. Let  $\omega(x) \in \Phi_\gamma^\beta$ ,  $0 \leq \beta < \gamma$ . Then

$$h^\beta \int_0^h \frac{\omega(x)}{x^{1+\beta}} dx \sim h^\gamma \int_h^\ell \frac{\omega(x)}{x^{1+\gamma}} dx \sim \omega(h), \quad (2.5)$$

on any subinterval  $[0, \ell - \delta]$ ,  $\delta > 0$ .

**P r o o f.** This statement is known, see [5] where  $\beta = 0$ , the proof is direct; we give the proof for completeness. By  $(\mathbb{Z}^\beta)$  and  $(\mathbb{Z}_\gamma)$ , it suffices to prove the inverse inequalities. Since the function  $\frac{\omega(x)}{x^\gamma}$  is almost decreasing, we have  $h^\beta \int_0^h \frac{\omega(x)}{x^{1+\beta}} dx \geq ch^\beta \frac{\omega(h)}{h^\gamma} \int_0^h x^{\gamma-\beta-1} dx = c\omega(h)$ . Similarly, since the function  $\frac{\omega(x)}{x^\beta}$  is almost increasing, we obtain  $h^\gamma \int_h^\ell \frac{\omega(x)}{x^{1+\gamma}} dx \geq ch^\gamma \frac{\omega(h)}{h^\beta} \int_h^\ell x^{-\gamma+\beta-1} dx \geq c\omega(h)$ , in case  $h$  is not allowed to approach the end-point  $\ell$ . ■

### 2.3. Generalized Bari, Lozinskii and Stechkin conditions

Let  $\beta \geq 0, \gamma > 0$ . For functions  $\omega \in W$  we consider the following well known conditions (see [1], where such conditions were treated for  $\beta = 0$  and in the case of increasing functions  $\omega$  in  $W$ ):

$$\sum_{k=n+1}^{\infty} k^{\beta-1} \omega\left(\frac{1}{k}\right) \leq cn^\beta \omega\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty, \quad c > 0, \quad (\mathbb{B}^\beta)$$

$$\sum_{k=1}^n k^{\gamma-1} \omega\left(\frac{1}{k}\right) \leq cn^\gamma \omega\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty, \quad c > 0 \quad (\mathbb{B}_\gamma)$$

(Bari type conditions),

$$\text{there exists a } C > 1 \text{ such that } \liminf_{h \rightarrow 0} \frac{\omega(Ch)}{\omega(h)} > C^\beta, \quad (\mathbb{L}^\beta)$$

$$\text{there exists a } C > 1 \text{ such that } \limsup_{h \rightarrow 0} \frac{\omega(Ch)}{\omega(h)} < C^\gamma, \quad (\mathbb{L}_\gamma)$$

(Lozinski type conditions),

$$\text{there exists a } \delta > 0 \text{ such that } \frac{\omega(t)}{t^{\beta+\delta}} \text{ is almost increasing, } (\mathbb{S}^\beta),$$

there exists a  $\delta > 0$  such that  $\frac{\omega(t)}{t^{\gamma-\delta}}$  is almost decreasing,  $(\mathbb{S}_\gamma)$ ,  
(Stechkin type conditions) and

for any  $\theta \in (0, 1)$  there exists an integer  $p = p(\theta)$  such that

$$p^\beta \omega\left(\frac{\ell}{pn}\right) < \theta \omega\left(\frac{\ell}{n}\right), \quad (\mathbb{P}^\beta)$$

for any  $\theta \in (0, 1)$  there exists an integer  $p = p(\theta)$  such that

$$\theta p^\gamma \omega\left(\frac{\ell}{pn}\right) > \omega\left(\frac{\ell}{n}\right). \quad (\mathbb{P}_\gamma)$$

### 3. Characterization of functions $\omega \in \Phi_\gamma^\beta$ in terms of the indices $m_\omega$ and $M_\omega$

**THEOREM 3.1.** Let  $\omega \in W$  and  $\beta \geq 0$ . Then:

- I)** condition  $(\mathbb{L}^\beta)$  is equivalent to the inequality  $m_\omega > \beta$ ;
- II)** conditions  $(\mathbb{B}^\beta), (\mathbb{L}^\beta), (\mathbb{Z}^\beta), (\mathbb{S}^\beta), (\mathbb{P}^\beta)$  are equivalent to each other: from the validity of one of them there follows the validity of all the others;
- III)** condition  $(\mathbb{S}^\beta)$  holds with any  $\delta < m_\omega - \beta$ .

**P r o o f.** We take  $\ell = 1$  for simplicity.

$$\begin{aligned} \text{I). The proof of part I) is direct: } m_\omega > \beta &\iff \sup_{x>1} \frac{\ln \left[ \liminf_{h \rightarrow 0} \frac{\omega(xh)}{\omega(h)} \right]}{\ln x} > \\ \beta &\iff \exists C > 1 : \frac{\ln \left[ \liminf_{h \rightarrow 0} \frac{\omega(Ch)}{\omega(h)} \right]}{\ln C} > \beta \iff \liminf_{h \rightarrow 0} \frac{\omega(Ch)}{\omega(h)} > C^\beta \iff (\mathbb{L}^\beta). \end{aligned}$$

For part **II)** we prove the following chain

$$(\mathbb{B}^\beta) \implies (\mathbb{Z}^\beta) \implies (\mathbb{L}^\beta) \implies (\mathbb{S}^\beta) \implies (\mathbb{P}^\beta) \implies (\mathbb{B}^\beta).$$

We suppose that  $\beta > 0$ , modifications for the case  $\beta = 0$  are easy: power functions should be replaced by the logarithmic function under the corresponding integration. We take  $\ell = 1$  without losing generality.

**The implication  $(\mathbb{B}^\beta) \implies (\mathbb{Z}^\beta)$ .** Let  $n = \left[\frac{1}{h}\right]$ ,  $\frac{1}{n+1} < h \leq \frac{1}{n}$  where  $h \in (0, 1]$ . The inequality is valid

$$\int_0^h \frac{\omega(t)}{t^{1+\beta}} dt \leq c_\omega \left[ c(\beta) \sum_{k=n+2}^{\infty} k^{\beta-1} \omega\left(\frac{1}{k}\right) + d(\beta) h^{1-\beta} \omega(h) \right], \quad (3.1)$$

where  $c(\beta) = 1$  when  $\beta \leq 1$  and  $c(\beta) = \frac{2^\beta - 1}{\beta}$  when  $\beta \geq 1$ , and  $d(\beta) = 2$  when  $\beta \leq 1$  and  $d(\beta) = 2 \cdot 3^{\beta-1}$  when  $\beta \geq 1$ . Indeed,

$$\begin{aligned} \int_0^h \frac{\omega(t)}{t^{1+\beta}} dt &= \sum_{k=n+2}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{\omega(t)}{t^{1+\beta}} dt + \int_{\frac{1}{n+2}}^h \frac{\omega(t)}{t^{1+\beta}} dt \\ &\leq c_\omega \sum_{k=n+2}^{\infty} \omega\left(\frac{1}{k}\right) \frac{(k+1)^\beta - k^\beta}{\beta} + c_\omega \omega(h) \frac{(n+2)^\beta - h^{-\beta}}{\beta}. \end{aligned}$$

Since  $(k+1)^\beta - k^\beta \leq ck^{\beta-1}$ , where  $c = \beta$  when  $\beta \leq 1$  and  $c = 2^\beta - 1$  when  $\beta \geq 1$ , and  $(n+2)^\beta \leq (1+2h)^\beta h^{-\beta}$ , we obtain (3.1).

From (3.1) we get

$$\begin{aligned} \frac{h^\beta}{\omega(h)} \int_0^h \frac{\omega(t)}{t} dt &\leq c_\omega \left[ c(\beta) \frac{h^\beta}{\omega(h)} \sum_{k=n+2}^{\infty} k^{\beta-1} \omega\left(\frac{1}{k}\right) + d(\beta) h^{1-\beta} \right] \\ &\leq c_\omega \left[ c(\beta) \frac{2^\beta c_\omega}{(n+1)^\beta \omega\left(\frac{1}{n+1}\right)} \sum_{k=n+2}^{\infty} k^{\beta-1} \omega\left(\frac{1}{k}\right) + d(\beta) h^{1-\beta} \right], \quad (3.2) \end{aligned}$$

whence the validity of  $(\mathbb{Z}^\beta)$  follows by the validity of  $(\mathbb{B}^\beta)$ .

**The implication  $(\mathbb{Z}^\beta) \implies (\mathbb{L}^\beta)$ .** Given  $(\mathbb{Z}^\beta)$ , that is,

$$\int_0^\delta \frac{\omega(t) dt}{t^{1+\beta}} \leq A \frac{\omega(\delta)}{\delta^\beta}, \quad 0 < \delta \leq 1, \quad (3.3)$$

we shall show that then

$$\frac{\omega(\xi)}{\xi^\beta} \leq M \frac{\omega(\delta)}{\delta^\beta}, \quad M = \frac{\beta 2^\beta A c_\omega}{2^\beta - 1} \quad (3.4)$$

for all  $\xi > 0, \delta > 0$  such that  $\frac{\xi}{\delta} \leq \frac{1}{2}$  ("ersatz" of the almost monotonicity).

Indeed, from (3.3) it follows that  $\int_\xi^\delta \frac{\omega(t) dt}{t^{1+\beta}} \leq A \frac{\omega(\delta)}{\delta^\beta}$  whence  $\frac{\omega(\xi)}{c_\omega} \int_\xi^\delta \frac{dt}{t^{1+\beta}} \leq A \frac{\omega(\delta)}{\delta^\beta}$ . Hence  $\frac{\omega(\xi)}{\xi^\beta} \cdot \frac{1 - (\frac{\xi}{\delta})^\beta}{\beta} \leq A c_\omega \frac{\omega(\delta)}{\delta^\beta}$  which yields (3.4) since  $\frac{\xi}{\delta} \leq \frac{1}{2}$ .



Now the key moment is that we repeat the same idea once more. For any  $\xi > 2\eta$  from (3.3) we obtain  $\int_{2\eta}^{\xi} \frac{\omega(t) dt}{t^{1+\beta}} \leq A \frac{\omega(\xi)}{\xi^\beta}$ . Then  $\frac{1}{M} \frac{\omega(\eta)}{\eta^\beta} \int_{2\eta}^{\xi} \frac{dt}{t} \leq A \frac{\omega(\xi)}{\xi^\beta}$  by (3.4) or

$$\frac{\omega(\eta)}{\eta^\beta} \leq \frac{AM}{\ln \frac{\xi}{2\eta}} \cdot \frac{\omega(\xi)}{\xi^\beta}. \quad (3.5)$$

We choose now a relation between  $\eta$  and  $\xi$  in the following way:  $\frac{\xi}{\eta} = C := 2e^{2AM} (> 2)$ . Then  $\frac{AM}{\ln \frac{\xi}{2\eta}} = \frac{1}{2}$  and from (3.5) we obtain  $\frac{\omega(\eta)}{\eta^\beta} \leq \frac{1}{2} \frac{\omega(C\eta)}{(C\eta)^\beta}$  for all  $\eta$  sufficiently small, that is,  $\omega(C\eta) \geq 2C^\beta \omega(\eta)$ . Hence  $\liminf_{\eta \rightarrow 0} \frac{\omega(C\eta)}{\omega(\eta)} \geq 2C^\beta > C^\beta$ , that is,  $(\mathbb{L}^\beta)$  has been obtained.

**The implication  $(\mathbb{L}^\beta) \implies (\mathbb{S}^\beta)$ .** Let  $(\mathbb{L}^\beta)$  be valid which means that there exists a  $C > 1$  such that

$$\nu = \nu(C) := \liminf_{h \rightarrow 0} \frac{\omega(Ch)}{C^\beta \omega(h)} > 1.$$

We have to show that  $\frac{\omega(x)}{x^{\beta+\delta}}$  is almost increasing for some  $\delta > 0$ . It suffices to prove this in a neighborhood of the origin. Let  $\nu_\varepsilon = \nu - \varepsilon$ ,  $0 < \varepsilon < \nu - 1$ . Then

$$\frac{\omega(Ch)}{C^\beta \omega(h)} \geq \nu - \varepsilon \quad \text{for} \quad 0 < h \leq h_0, \quad h_0 = h_0(\varepsilon). \quad (3.6)$$

We choose  $\delta = \delta(\varepsilon) = \frac{\ln \nu_\varepsilon}{\ln C} > 0$  and show that  $\frac{\omega(x)}{x^{\beta+\delta}}$  is almost increasing under this choice of  $\delta$ .

For  $0 < h \leq h_0$  according to (3.6) we have

$$\frac{\omega(h)}{h^{\beta+\delta}} \leq \frac{\omega(Ch)}{(Ch)^{\beta+\delta}}, \quad 0 < h \leq h_0. \quad (3.7)$$

Now, for arbitrary  $0 < h_1 < h \leq h_0$  we choose an integer  $N$  by the condition  $C^N h_1 \leq h < C^{N+1} h_1$  ( $N = \left\lceil \log_C \frac{h}{h_1} \right\rceil$ ). Then by (3.7) we get

$$\frac{\omega(h_1)}{h_1^{\beta+\delta}} \leq \frac{\omega(Ch_1)}{(Ch_1)^{\beta+\delta}} \leq \dots \leq \frac{\omega(C^N h_1)}{(C^N h_1)^{\beta+\delta}}.$$

Since  $\omega(h)$  is almost increasing, we obtain  $\frac{\omega(h_1)}{h_1^{\beta+\delta}} \leq c_\omega C^{\beta+\delta} \frac{\omega(h)}{(C^{N+1} h_1)^{\beta+\delta}} \leq c_\omega C^{\beta+\delta} \frac{\omega(h)}{h^{\beta+\delta}}$  which means that  $\frac{\omega(h)}{h^{\beta+\delta}}$  is almost increasing. This fact has been

proved for any  $0 < \delta = \frac{\ln \nu_\varepsilon}{\ln C} = \frac{\ln \nu}{\ln C} - \varepsilon_1 = \frac{\ln \left( \liminf_{h \rightarrow 0} \frac{\omega(Ch)}{\omega(h)} \right)}{\ln C} - \beta - \varepsilon_1$  with an arbitrarily small  $\varepsilon_1$ . Therefore, the function  $\frac{\omega(x)}{x^{\beta+\delta}}$  is almost increasing for any

$$0 < \delta < \sup_{C>1} \frac{\ln \left( \liminf_{h \rightarrow 0} \frac{\omega(Ch)}{\omega(h)} \right)}{\ln C} - \beta = m_\omega - \beta. \quad (3.8)$$

**The implication  $(\mathbb{S}^\beta) \implies (\mathbb{P}^\beta)$ .** Let  $\frac{\omega(x)}{x^{\beta+\delta}}$  be almost increasing for some  $\delta > 0$ :  $\frac{\omega(x)}{x^{\beta+\delta}} \leq B \frac{\omega(y)}{y^{\beta+\delta}}$  for  $0 < x \leq y \leq 1$ . We here choose  $x = \frac{1}{pn}$  and  $y = \frac{1}{n}$  and obtain  $p^{\beta+\delta} \omega\left(\frac{1}{pn}\right) \leq B \omega\left(\frac{1}{n}\right)$ ,  $n = 1, 2, 3, \dots$  where the integer  $p$  is to be chosen. Given an arbitrary  $\theta \in (0, 1)$ , we choose  $p = p(\theta) > \left(\frac{B}{\theta}\right)^{\frac{1}{\delta}}$  such that  $p^\delta > \frac{B}{\theta}$  and then  $p^\beta \omega\left(\frac{1}{pn}\right) < \theta \omega\left(\frac{1}{n}\right)$ .

**The implication  $(\mathbb{P}^\beta) \implies (\mathbb{B}^\beta)$ .** We have

$$\sum_{k=n+1}^{\infty} k^{\beta-1} \omega\left(\frac{1}{k}\right) = \sum_{s=0}^{\infty} \sum_{k=p^s n+1}^{p^{s+1}n} k^{\beta-1} \omega\left(\frac{1}{k}\right)$$

for any choice of the integer  $p$ . Since the function  $\omega$  is almost increasing, we then get

$$\sum_{k=n+1}^{\infty} k^{\beta-1} \omega\left(\frac{1}{k}\right) \leq c_\omega \sum_{s=0}^{\infty} \omega\left(\frac{1}{p^s n}\right) \sum_{k=p^s n+1}^{p^{s+1}n} k^{\beta-1}.$$

Obviously,  $\sum_{k=m}^n k^{\beta-1} = \sum_{k=m}^n k^{\beta-1} \int_{k-1}^k dx \leq c \sum_{k=m}^n \int_{k-1}^k x^{\beta-1} dx$  with some constant  $c > 0$  for all  $k \geq 2$  ( $c = 1$  when  $\beta \leq 1$ ). Therefore,  $\sum_{k=m}^n k^{\beta-1} \leq c \int_{m-1}^n x^{\beta-1} = c \frac{n^\beta - (m-1)^\beta}{\beta}$  and then

$$\sum_{k=n+1}^{\infty} k^{\beta-1} \omega\left(\frac{1}{k}\right) \leq c c_\omega \frac{n^\beta (p^\beta - 1)}{\beta} \sum_{s=0}^{\infty} p^{\beta s} \omega\left(\frac{1}{p^s n}\right).$$

By condition  $(\mathbb{P}^\beta)$ , for any  $\theta \in (0, 1)$  we can choose an integer  $p$  such that  $\omega\left(\frac{1}{p^s n}\right) \leq \left(\frac{\theta}{p^\beta}\right)^s \omega\left(\frac{1}{n}\right)$ . Consequently,

$$\sum_{k=n+1}^{\infty} k^{\beta-1} \omega\left(\frac{1}{k}\right) \leq c_1 n^\beta \omega\left(\frac{1}{n}\right) \sum_{s=0}^{\infty} \theta^s,$$

with  $c_1 = \frac{cc_\omega}{\beta}(p^\beta - 1)$ , that is,  $(\mathbb{B}^\beta)$  holds.

**III).** This part was already proved in (3.8). ■

**THEOREM 3.2.** *Let  $\omega \in W$  and  $\gamma > 0$ . Then:*

- I)** *condition  $(\mathbb{L}_\gamma)$  is equivalent to the inequality  $\gamma > M_\omega$ ;*
- II)** *conditions  $(\mathbb{B}_\gamma), (\mathbb{L}_\gamma), (\mathbb{Z}_\gamma), (\mathbb{S}_\gamma), (\mathbb{P}_\gamma)$  are equivalent to each other: from the validity of one of them there follows the validity of all the others;*
- III)** *condition  $(\mathbb{S}_\gamma)$  holds with any  $\delta < \gamma - M_\omega$ .*

The proof of Theorem 3.2 is a counterpart of that of Theorem 3.1 and thereby is omitted.

**REMARK 3.3.** Statement **II** of Theorems 3.1 and 3.2 was proved in [1] in the case when  $\beta = 0$  and functions  $\omega$  are monotonic. We followed mainly the ideas of the proof in [1], with modifications everywhere, where the arguments from [1] were not valid for almost increasing functions. Theorems 3.1 and 3.2 for almost increasing functions  $\omega$  were earlier proved for  $\beta = 0$  and  $\gamma = 1$  in [21]. Statements of Theorems 3.1 and 3.2 for almost increasing functions and arbitrary  $0 \leq \beta < \gamma < \infty$  were considered in [18], but the proof there was not complete.

**COROLLARY 3.4.** *The class  $\Phi_\gamma^\beta$ ,  $\beta \geq 0, \gamma > 0$  is non-empty if and only if  $\beta < \gamma$ .*

**P r o o f.** Indeed, from statement **I** of Theorems 3.1 and 3.2, it follows that if  $\omega \in \Phi_\gamma^\beta$ , then  $m_\omega > \beta$  and  $M_\omega < \gamma$ . But  $m_\omega \leq M_\omega$ , so that  $\beta < \gamma$ . Inversely, if  $\beta < \gamma$ , then the power function  $x^a$  with  $\beta < a < \gamma$  obviously belongs to  $\Phi_\gamma^\beta$ . ■

**THEOREM 3.5.** *A function  $\omega \in W$  belongs to  $\mathcal{Z}^\beta$ ,  $\beta \geq 0$ , if and only if  $m_\omega > \beta$  and it belongs to  $\mathcal{Z}_\gamma$ ,  $\gamma > 0$ , if and only if  $M_\omega < \gamma$ , so that in the case  $0 \leq \beta < \gamma$*

$$\omega \in \Phi_\gamma^\beta \iff \beta < m_\omega \leq M_\omega < \gamma \quad (3.9)$$

and for  $\omega \in \Phi_\gamma^\beta$  and any  $\varepsilon > 0$  there exist constants  $c_1 = c_1(\varepsilon) > 0$  and  $c_2 = c_2(\varepsilon) > 0$  such that

$$c_1 x^{M_\omega + \varepsilon} \leq \omega(x) \leq c_2 x^{m_\omega - \varepsilon}, \quad 0 \leq x \leq \ell. \quad (3.10)$$

**P r o o f.** Indeed, according to statements **II** and **I** of Theorem 3.1,  $\omega \in \mathcal{Z}^\beta \iff m_\omega > \beta$ . Similarly, by Theorem 3.2,  $\omega \in \mathcal{Z}_\gamma \iff M_\omega < \gamma$ . Therefore,  $\omega \in \Phi_\gamma^\beta \iff \beta < m_\omega \leq M_\omega < \gamma$ .

To get at inequalities (3.10), it suffices to observe that the function  $\frac{\omega(x)}{x^{m_\omega - \varepsilon}}$  is almost increasing and the function  $\frac{\omega(x)}{x^{M_\omega + \varepsilon}}$  is almost decreasing for any  $\varepsilon > 0$  according to statement **III** of Theorems and , and any almost increasing and almost decreasing function is bounded from above and from below, respectively. ■

The following theorem characterizes the conditions  $(\mathbb{S}^\beta)$  and  $(\mathbb{S}_\gamma)$  in terms of the indices  $m_\omega$  and  $M_\omega$ .

**THEOREM 3.6.** *For any function  $\omega \in \mathcal{Z}^\beta$  its lower index  $m_\omega$  may be calculated by the formula*

$$m_\omega = \sup \left\{ \delta > \beta : \frac{\omega(x)}{x^\delta} \text{ is almost increasing} \right\}, \quad (3.11)$$

while for any  $\omega \in \mathcal{Z}_\gamma$  its upper index  $M_\omega$  is calculated by the formula

$$M_\omega = \inf \left\{ \delta \in (0, \gamma) : \frac{\omega(x)}{x^\delta} \text{ is almost decreasing} \right\}. \quad (3.12)$$

**P r o o f.** Let  $a = \sup \left\{ \delta > \beta : \frac{\omega(x)}{x^\delta} \text{ is almost increasing} \right\}$ . By statement **III**) of Theorem 3.1,  $m_\omega \leq a$ . We have to prove that  $m_\omega = a$ . Suppose to the contrary that  $m_\omega < a$ . Then the function  $\omega_1(x) = \frac{\omega(x)}{x^{m_\omega}}$  is also almost increasing and  $\omega_1(0) = 0$  since the function  $\frac{\omega(x)}{x^{m_\omega + \delta}}$  is also almost increasing with  $0 < \delta < a - m_\omega$ . Therefore,  $\omega_1 \in W$  and  $\omega_1(x)$  satisfies condition  $\mathbb{S}^0 = \mathbb{S}^\beta|_{\beta=0}$ . Then, by statement **I**) of Theorem 3.1,  $m_{\omega_1} > 0$ , which is impossible since  $m_{\omega_1} = m_\omega - m_\omega = 0$ .

Similarly, formula (3.12) is obtained. ■

## 4. One-dimensional integral operators in the space $H_0^\omega(\Omega, \rho)$

### 4.1. Fractional integration operators

Let  $\Omega = [a, b]$ ,  $-\infty < a < b < \infty$  and

$$H^\omega(\Omega) = \{f(x) : \omega(f, h) \leq c\omega(h), \quad 0 < h < \ell = b - a\},$$

where  $\omega(f, h) = \max_{\substack{x \in \Omega, y \in \Omega \\ |x-y| \leq h}} |f(x) - f(y)|$ . The function  $\omega(h)$ , referred to in the sequel as *the characteristic function of the space, or characteristics*, will be supposed to belong to the Zygmund-Bari-Stechkin class  $\Phi$ .

Let  $\Pi = \{a_0, a_1, \dots, a_n\}$  be any finite set of points on  $\Omega$  and  $\rho(x)$  any non-negative function on  $\Omega$  vanishing only at the points of the set  $\Pi$ . We define the space  $H_0^\omega(\Omega, \rho)$  as

$$H_0^\omega(\Omega, \rho) = \left\{ f(x) : \rho(x)f(x) \in H^\omega(\Omega), \lim_{x \rightarrow a_k} [\rho(x)f(x)] = 0, \right. \\ \left. a_k \in \Pi, \quad k = 0, 1, \dots, n \right\}. \quad (4.1)$$

Equipped with the norm

$$\|f\|_{H_0^\omega(\Omega, \rho)} = \|\rho f\|_{H_0^\omega(\Omega)} = \|\rho f\|_{C(\Omega)} + \sup_{h>0} \frac{\omega(\rho f, h)}{\omega(h)},$$

this is a Banach space. (The first term in this norm is essential in the case of the space  $H^\omega(\Omega)$ , on functions in  $H_0^\omega(\Omega)$  it may be omitted).

When considering the Riemann-Liouville fractional integration operator  $I_{a+}^\alpha$ , we admit complex values of  $\alpha$ , including the purely imaginary orders with  $\operatorname{Re} \alpha = 0$ . In the case  $\operatorname{Re} \alpha = 0$  we follow the known modification of definition of the fractional integral via the Marchaud formula ([24], p.225):

$$I_{a+}^{i\theta} f(x) = \frac{f(x)(x-a)^{i\theta}}{\Gamma(1+i\theta)} - \frac{1}{\Gamma(i\theta)} \int_a^x \frac{f(x) - f(t)}{(x-t)^{1+i\theta}} dt, \quad \theta \in \mathbb{R}^1, \quad (4.2)$$

see [7], p.182.

**a). The case of power weight.**

We consider the power weight of the form

$$\rho(x) = \prod_{k=0}^n |x - a_k|^{\nu_k}, \quad (4.3)$$

where

$$a_k \in \Pi = \{a_0, a_1, a_2, \dots, a_{n-1}, a_n\} \subset [a, b], \quad a_0 = a, \quad a_n = b, \quad n \geq 1. \quad (4.4)$$

**THEOREM 4.1.** *Let  $\rho$  be the power weight of form (4.3). The Riemann-Liouville fractional integration operator  $I_{a+}^\alpha$  with  $0 \leq \operatorname{Re} \alpha < 1$ , maps boundedly the space  $H_0^\omega(\Omega, \rho)$  onto the space  $H_0^{\omega_\alpha}(\Omega, \rho)$ :*

$$I_{a+}^\alpha [H_0^\omega(\Omega, \rho)] = H_0^{\omega_\alpha}(\Omega, \rho), \quad (4.5)$$

where  $\omega \in W$  and  $\omega_\alpha(h) = h^{\operatorname{Re} \alpha} \omega(h)$ , if

$$0 < m_\omega \leq M_\omega < 1 - \operatorname{Re} \alpha, \quad (4.6)$$

$$\nu_0 < 1 + m_\omega \quad \text{and} \quad \nu_n = 0 \quad \text{or} \quad \nu_n > M_\omega + \operatorname{Re} \alpha, \quad (4.7)$$

and

$$M_\omega + \operatorname{Re} \alpha < \nu_k < 1 + m_\omega, \quad k = 1, 2, \dots, n-1. \quad (4.8)$$

**P r o o f.** The statement (4.5) is known under the conditions

$$\nu_0 < 2 - \operatorname{Re} \alpha, \quad \nu_n = 0 \quad \text{or} \quad \nu_n > \operatorname{Re} \alpha \quad (4.9)$$

and

$$\operatorname{Re} \alpha < \nu_k < 2 - \operatorname{Re} \alpha, \quad k = 1, 2, \dots, n-1, \quad (4.10)$$

and Zygmund-type conditions on  $\omega$ :  $\omega \in \Phi_\gamma^\beta$ , where

$$\beta = \max(1, \nu_0, \nu_1, \dots, \nu_n) - 1, \quad \gamma = \begin{cases} \min(1, \nu_1, \dots, \nu_{n-1}) - \operatorname{Re} \alpha, & \nu_n = 0 \\ \min(1, \nu_1, \dots, \nu_n) - \operatorname{Re} \alpha, & \nu_n > \operatorname{Re} \alpha. \end{cases} \quad (4.11)$$

(Under conditions (4.9)-(4.11) relation (4.5) for real  $\alpha \in (0, 1)$  was proved in [15] in the non-weighted case, see [24], p. 253; in [16], [25], see [24], p. 254, for the case of the power weight (4.3) with  $n = 0$ ,  $\Pi = \{a\}$  or  $n = 1$ ,  $\Pi = \{a, b\}$  (clearly, one should write  $\beta = \max(1, \nu_0)$  and  $\gamma = 1 - \Re \alpha$  in the case  $n = 0$ .) The case of complex  $\alpha$  including the purely imaginary orders and arbitrary  $n$  and  $\Pi$  was treated in [8]). Making use of Theorem 3.5, we observe that condition (4.11) is equivalent to the inequalities

$$\max(1, \nu_0, \nu_1, \dots, \nu_n) - 1 < m_\omega, \quad M_\omega < \gamma. \quad (4.12)$$

Since  $m_\omega \leq M_\omega$ , it is easily seen that the set of conditions (4.6) and (4.7) is the same as the set of conditions (4.9), (4.10) and (4.12). ■

**REMARK 4.2.** In case  $\Pi = \{a, b\}$ , a statement very close to Theorem 4.1 was obtained in [35] (see Theorem 3.3 in [35]) in different terms, imitating in a certain sense, the information about the indices  $m_\omega$  and  $M_\omega$ .

**REMARK 4.3.** In the non-weighted case, that is,  $\Pi = \{a\}$  and  $\nu_0 = 0$  there remain only conditions (4.6). It is worthwhile noticing that the one way statement  $I_{a+}^\alpha : H_0^\omega(\Omega) \longrightarrow H_0^{\omega\alpha}(\Omega)$  requires only one of the inequalities in (4.6), namely,  $M_\omega + \operatorname{Re} \alpha < 1$ , see inequality (13.62) in [24], while

the inverse statement  $(I_{a+}^\alpha)^{-1} : H_0^{\omega_\alpha}(\Omega) \longrightarrow H_0^\omega(\Omega)$ , on the contrary, requires only the condition  $m_\omega > 0$ , see (13.66) in [24]. Thus, the fractional integration operator  $I_{a+}^\alpha$  is bounded for example, from  $H_0^\omega(\Omega)$  with  $\omega(x) = \frac{1}{\ln \frac{m}{x}}$ ,  $m > b-a$ , into  $H_0^{\omega_\alpha}(\Omega)$  with  $\omega_\alpha(x) = \frac{x^\alpha}{\ln \frac{m}{x}}$ , but we are already unable to state that the range  $I_{a+}^\alpha[H_0^\omega(\Omega)]$  coincides with the whole space  $H_0^{\omega_\alpha}(\Omega)$  in this case.

**b). The case of weight more general than a power one.**

A statement analogous to Theorem 4.1 is also valid for weights more general than power weights (4.3), but having a similar structure of a product of almost increasing functions fixed to a finite number of points on  $[a, b]$ . For simplicity we deal with the case when the weight is fixed only to the initial point  $x = a$ .

**DEFINITION 4.4.** A function  $\psi \in W$  is said to belong to  $W_\mu$ , if  $\frac{\psi(x)}{x^\mu}$  is almost decreasing and  $\psi(x)$  satisfies condition

$$\left| \frac{\psi(x) - \psi(y)}{x - y} \right| \leq c \frac{\psi(x^*)}{x^*}, \quad x^* = \max(x, y), \quad c > 0. \quad (4.13)$$

Observe that condition (4.13) in Definition 4.4 is satisfied automatically, if  $\frac{\psi(x)}{x^\mu}$  is decreasing (instead of being almost decreasing).

**THEOREM 4.5.** Let  $\rho = \psi(x - a)$ , where  $\psi \in W_\mu, \mu > 0$ . The Riemann-Liouville fractional integration operator  $I_{a+}^\alpha$  with  $0 < \alpha < 1$ , maps boundedly the space  $H_0^\omega(\Omega, \rho)$  with  $\omega \in W$  onto the space  $H_0^{\omega_\alpha}(\Omega, \rho)$  with  $\omega_\alpha(h) = h^\alpha \omega(h)$ :

$$I_{a+}^\alpha [H_0^\omega(\Omega, \rho)] = H_0^{\omega_\alpha}(\Omega, \rho), \quad (4.14)$$

if

$$0 < m_\omega \leq M_\omega < 1 - \alpha \quad \text{and} \quad 0 < \mu < 1 + m_\omega. \quad (4.15)$$

**P r o o f.** The isomorphism (4.14) was proved in [26], Th. 6, under the assumptions

$$0 < \mu < 2 - \alpha \quad \text{and} \quad \omega \in \Phi_\gamma^\beta, \quad \text{with} \quad \beta = \max(1, \mu) - 1, \quad \gamma = 1 - \alpha. \quad (4.16)$$

Now, having proved Theorem 3.5, we can reformulate the condition  $\omega \in \Phi_{1-\alpha}^{\max(1, \mu)-1}$  in the form (4.6), the assumption  $0 < \mu < 2 - \alpha$  being satisfied automatically, since  $\mu < 1 + m_\omega \leq 1 + M_\omega < 2 - \alpha$ . ■

REMARK 4.6. Observe that conditions imposed on the weight function admit oscillating weights  $\psi(x)$ , in particular, oscillating between two power functions with different exponents.

**c). The periodic case.**

For the Weyl fractional integration of periodic functions ([24], Sect. 19)

$$I_+^{(\alpha)}\varphi(x) := \frac{1}{2\pi} \int_0^{2\pi} \Psi^\alpha(t) \varphi(x-t) dt = \frac{1}{\Gamma(\alpha)} \lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{Z}^+}} \int_{x-2n\pi}^x \frac{\varphi(t) dt}{(x-t)^{1-\alpha}}, \quad (4.17)$$

where  $\Psi^\alpha(x) = \sum'_{n \in \mathbb{Z}} \frac{e^{inx}}{(in)^\alpha} = \sum_{n=1}^{\infty} \frac{\cos(nx - \frac{\alpha\pi}{2})}{n^\alpha}$  a result similar to Theorem 4.1 with the periodic power weight may be proved. We mention here a result for a non-weighted case, but for the spaces more general than  $H^\omega$ .

Let

$$\Delta_h^k f(x) := \sum_{j=0}^k (-1)^j \binom{k}{j} f(x-jh)$$

be a finite difference of a periodic function  $f(x)$ .

DEFINITION 4.7. By  $H_p^{\omega,k} = H_p^{\omega,k}(0, 2\pi)$  we denote the space of those functions in  $L_p(0, 2\pi)$  for which  $\sup_{h>0} \frac{\|\Delta_h^k f\|_p}{|h|^k} < \infty$ ,  $1 \leq p \leq \infty$ .

$$\text{Let } \bar{\alpha} = \begin{cases} \alpha, & \text{if } \alpha \text{ is not an integer} \\ \alpha + 1, & \text{if } \alpha \text{ is an integer} \end{cases}.$$

THEOREM 4.8. Let  $s = 1, 2, \dots$ ,  $1 \leq p \leq \infty$  and  $\alpha > 0$ . The following isomorphism holds:

$$I_+^{(\alpha)}(H_p^{\omega,s}) = H_p^{\omega_\alpha,s}, \quad \omega_\alpha(x) = x^\alpha \omega(x), \quad (4.18)$$

if  $s > \bar{\alpha}$  and  $0 < m_\omega \leq M_\omega < s - \bar{\alpha}$ .

P r o o f. The statement (4.18) was proved in [9] under the assumptions that  $s > \bar{\alpha}$  and  $\omega$  belongs to the class

$$A^{\lambda,\mu} := \left\{ \omega \in W : \frac{\omega(x)}{x^\lambda} \text{ is almost increasing, } \frac{\omega(x)}{x^\mu} \text{ is almost decreasing} \right\}$$

with  $0 < \lambda < \mu \leq s - \bar{\alpha}$ . By Theorems 3.5 and 3.6, we have

$$\Phi_{\mu-\varepsilon}^{\lambda+\varepsilon} \subset A^{\lambda,\mu} \subset \Phi_\mu^\lambda$$



for all  $0 \leq \lambda < \mu < \infty$  and arbitrarily small  $\varepsilon \in \left(0, \frac{\mu-\lambda}{2}\right)$ . Therefore, statement (4.18) is valid if  $\omega \in \Phi_\mu^\lambda$  with any  $\lambda$  and  $\mu$  such that  $0 < \lambda < \mu < s - \bar{\alpha}$ . Then by (3.9), the isomorphism in (4.18) holds if  $0 < m_\omega \leq M_\omega < s - \bar{\alpha}$  which proves the theorem. ■

## 4.2. More general operators

We consider the operators

$$Kf(x) := \int_a^x k(x-t)f(t) dt, \quad x \in [a, b], \quad -\infty < a < b < \infty \quad (4.19)$$

with kernels  $k(x)$  having singularity at the point  $x = 0$ , which include, in particular, the case of the power kernel  $k(x) = x^{\alpha-1}$ ,  $0 < \alpha < 1$ .

The class of kernels we will deal with is described by the following definition.

**DEFINITION 4.9.** A kernel  $k(x)$  is said to belong to the class  $V_\lambda$ ,  $\lambda > 0$ , if the following conditions are satisfied:

- i)  $k(x) \geq 0$ ,  $x \in (0, b-a)$ ;
- ii)  $x^\lambda k(x)$  is almost increasing and there exists an  $\varepsilon > 0$  such that  $x^{\lambda-\varepsilon} k(x)$  is almost decreasing;
- iii)  $k(x)$  satisfies the condition of the type (4.13):

$$\left| \frac{\psi(x+h) - \psi(x)}{h} \right| \leq c \frac{\psi(x)}{x+h}, \quad h > 0, \quad c > 0. \quad (4.20)$$

Observe that Condition (4.13) is well adjusted for almost increasing functions, while (4.20) suits well for almost decreasing functions.

In the case of a general non-power kernel it is difficult to expect to obtain mapping onto as in (4.14). Instead, we prove a statement on imbedding of the range  $K[H_0^\omega(\Omega, \rho)]$  into another space  $H_0^{\omega_k}(\Omega, \rho)$  where  $\omega_k(h) = hk(h)\omega(h)$ , see Theorem 4.10.

We also consider weights more general than power ones, but for simplicity deal with the case when the weight is fixed only to the initial point  $x = a$ .

**THEOREM 4.10.** Let  $k \in V_\lambda$ ,  $0 < \lambda < 1$ ,  $\rho = \psi(x-a)$ , where  $\psi \in W_\mu$ ,  $\mu > 0$ . The operator  $K$  is bounded from the space  $H_0^\omega(\Omega, \rho)$ ,  $\omega \in W$ , into the space  $H_0^{\omega_k}(\Omega, \rho)$ , where  $\omega_k(h) = hk(h)\omega(h)$ , if

$$m_\omega > 0, \quad M_{\omega_k} < 1 \quad \text{and} \quad \mu < 1 + \max(m_\omega, \lambda). \quad (4.21)$$

*P r o o f.* The statement of the theorem was proved in [26], Th. 3, under the assumption that

$$\mu < 1 + \lambda, \quad \omega \in \mathcal{Z}^{max(0, \mu-1)}, \quad xk(x)\omega(x) \in \mathcal{Z}_1.$$

Making use of Theorem 3.5, we have proved specially for this goal, we can recalculate these conditions in the form (4.21). ■

## 5. Multi-dimensional fractional operators in the space $H_0^\omega(\Omega, \rho)$

### 5.1. Spatial fractional integrals

Let  $\dot{\mathbb{R}}^n$  be a compactification of  $\mathbb{R}^n$  by the unique infinite point. Correspondingly, the generalized Hölder space  $H^\omega(\dot{\mathbb{R}}^n)$  is defined as

$$H^\omega(\dot{\mathbb{R}}^n) := \left\{ f \in C(\mathbb{R}^n) : \sup_{x, y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{\omega[d(x, y)]} < \infty \right\}, \quad (5.1)$$

where  $d(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}$  (observe that  $\sup_{x, y \in \mathbb{R}^n} \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} = 1$ ).

The weighted space is defined in the usual way:  $H^\omega(\dot{\mathbb{R}}^n, \rho) = \{f : \rho f \in H^\omega(\dot{\mathbb{R}}^n)\}$ .

For the Riesz fractional integral (1.4) in  $\mathbb{R}^n$  and a special weight fixed to infinity, the following statement is a reformulation of the results obtained in [29], [30].

**THEOREM 5.1.** *Let  $\omega \in W$ ,*

$$\omega_\alpha(h) = h^\alpha \omega(h) \quad \text{and} \quad \rho_\alpha(x) = (\sqrt{1+|x|^2})^{n+\alpha}.$$

*The operator  $I^\alpha$  with  $0 < \alpha < 1$ , maps boundedly the space  $H_0^\omega(\dot{\mathbb{R}}^n, \rho_\alpha)$  onto the space  $H_0^{\omega_\alpha}(\dot{\mathbb{R}}^n, \rho_{-\alpha})$ :*

$$I^\alpha \left[ H_0^\omega(\dot{\mathbb{R}}^n, \rho_\alpha) \right] = H_0^{\omega_\alpha}(\dot{\mathbb{R}}^n, \rho_{-\alpha}), \quad (5.2)$$

*if  $0 < m_\omega \leq M_\omega < 1 - \alpha$ .*

### 5.2. Spherical fractional integrals

The generalized Hölder space on the unit sphere  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$  is defined in the usual way. We take  $\rho(x) = |x - a|^\mu$  as the weight function,  $a \in S_{n-1}$ .

We consider the operator slightly more general than the spherical fractional integral (1.4):

$$K^{\alpha,\nu} f(x) = \int_{\mathbb{S}^{n-1}} \frac{\ln^\nu \frac{m}{|x-\sigma|}}{|x-\sigma|^{n-1-\alpha}} f(\sigma) d\sigma, \quad x \in \mathbb{S}^{n-1}, \quad m > 2. \quad (5.3)$$

For the operator  $K^{\alpha,\nu}$  the following statement is derived from the results in [33].

THEOREM 5.2. *Let  $\omega \in W$  and*

$$\omega_{\alpha,\nu}(h) = h^\alpha \left( \ln \frac{m}{x} \right)^\nu \omega(h).$$

*The operator  $K^{\alpha,\nu}$ , where  $0 < \operatorname{Re} \alpha < 1$  and  $\nu \in \mathbb{R}^1$ , is bounded from the space  $H_0^\omega(S^{n-1}, \rho)$  into the space  $H_0^{\omega_{\alpha,\nu}}(S^{n-1}, \rho)$ ,  $\rho(x) = |x-a|^\mu$ , if*

$$M_\omega + \operatorname{Re} \alpha < 1 \quad \text{and} \quad M_\omega + \operatorname{Re} \alpha < \mu < n - 1 + m_\omega. \quad (5.4)$$

P r o o f. The statement on boundedness of the operator  $K^{\alpha,\nu}$  was proved in [33] under the following assumptions:

- a)  $\omega \in \mathcal{Z}_{\min(1,\mu)-\operatorname{Re} \alpha}$  in the case  $\operatorname{Re} \alpha < \mu < n - 1$ ;
- b)  $\omega \in \Phi_{1-\operatorname{Re} \alpha}^{\mu+1-n}$  in the case  $n - 1 \leq \mu < n - \operatorname{Re} \alpha$ .

It is a matter of direct recalculation based on Theorem 3.5 to check that assumptions a),b) are equivalent to conditions (5.4). ■

REMARK 5.3. Observe, that in the "one way" statement of Theorem 5.2 there is admitted the situation when  $m_\omega = 0$ . This may happen if  $M_\omega + \operatorname{Re} \alpha < \mu < n - 1$ , see also Remark 4.3.

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